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Thermodynamic Properties of the One-Dimensional Two-Component Log-Gas

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We consider a one-dimensional continuum gas of pointlike positive and negative unit charges interacting via a logarithmic potential. The mapping onto a two-dimensional boundary sine-Gordon field theory with zero bulk mass provides the full thermodynamics (density-fugacity relationship, specific heat, etc.) of the log-gas in the whole stability range of inverse temperatures $\beta < 1$. An exact formula for the excess chemical potential of a "foreign" particle of an arbitrary charge, put into the log-gas, is derived. The results are checked by a small-β expansion and at the collapse $\beta = 1$ point. The possibility to go beyond the collapse temperature is discussed.

KEY WORDS: Two-component plasma; one dimension; logarithmic interaction; thermodynamics; boundary sine-Gordon model; integrability.

1. INTRODUCTION

The model under consideration is a two-dimensional (2D) two-component (TC) classical Coulomb gas confined to a 1D manifold, the circle of radius R or its $R \to \infty$ limit the straight line. It is usually named as the symmetric 1D TC log-gas. The model consists of a mixture of mobile pointlike particles $\{j\}$ of positive and negative, say unit, charges $\{q_j = \pm\}$, localized at continuous angle positions $\{\varphi_j\}$ on the circle of length $L = 2\pi R$. The interaction energy reads

$$E(\{q_j, \varphi_j\}) = \sum_{j<k} q_j q_k v(\varphi_j, \varphi_k) \tag{1.1}$$

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where $v(\varphi, \varphi')$ is the 2D Coulomb potential with imposed periodic boundary conditions,

$$v(\varphi, \varphi') = -\ln \left\{ \left(\frac{R}{r_0} \right) |e^{i\varphi} - e^{i\varphi'}| \right\} \tag{1.2}$$

The length constant r_0, which fixes the zero point of energy, will be set for simplicity to unity. In the thermodynamic limit $R \to \infty$, which is of interest in this paper, introducing the straight line position variable $x \in (-\infty, \infty)$ via $x = \varphi R$, the interaction potential (1.2) takes the familiar logarithmic form

$$v(x, x') = -\ln(|x - x'|) \tag{1.3}$$

For pointlike particles, the interaction Boltzmann factor of a positive-negative pair of charges at distance x, $x^{-\beta}$ where β is the inverse temperature, is integrable on a 1D manifold at small x if and only if $\beta < 1$ (at large x, there is no problem because the interaction is screened by the system), with $\beta = 1$ being the collapse point. In its conductive phase, the model exhibits poor screening properties, a typical feature of a D-dimensional Coulomb system confined to a domain of dimension $D-1$.[1] As a consequence, the leading non-oscillatory term of the large-distance asymptotic decay of the charge charge correlation is algebraically slow $\sim -1/[\beta(\pi x)^2]$.[2]

The 1D TC log-gas, with $\pm$ charges required to alternate in space, is equivalent to the impurity Kondo problem.[3–5]

The 1D TC log-gas, without any restriction on the charge order, is related to the problem of quantum Brownian motion of a particle in a 1D periodic potential.[6–8] The model was also studied in connection with the problem of non-equilibrium quantum transport through a point contact in a 1D Luttinger liquid,[9–11] having a realization in resonant tunneling-transport experiments between edge states in fractional quantum Hall effect devices.[12]

The 1D TC log-gas with a hard core, or with some other short-distance regularization of the Coulomb interaction potential, undergoes a 1D counterpart of the Kosterlitz–Thouless transition from a conducting phase to a low-temperature insulating phase around $\beta = 2$. An approximate analytic study of the gas with $\log(1+|x|)$ interaction was given in ref. 13. The lattice version of the model was exactly solved at $\beta = 1, 2, 4$ and its conductor-insulator phase diagram was conjectured in refs. 14–16.

Recently,[17] the bulk thermodynamic properties (density-fugacity relationship, free energy, internal energy, specific heat, etc.) of the infinite

2D TC plasma have been obtained exactly, in the whole model's stability range of temperatures, by mapping the plasma onto a 2D bulk sine-Gordon field theory and then using recent results about that integrable field theory. The aim of this paper is to derive the full thermodynamics of the confined plasma, the 1D TC log-gas, via its relationship to the 2D boundary sine-Gordon field theory with zero bulk mass. The particle nature of the statistical model permits one to check the obtained behaviour of thermodynamic quantities close to the collapse $\beta = 1$ point, and to suggest a possible analytic extension of the results beyond that collapse point.

The mapping onto the 2D boundary sine-Gordon model is outlined within the grandcanonical formalism in Section 2. To give to the mapping a precise meaning, we analyse the short-distance behaviour of the pair distribution function for two basic combinations of particle charges. The relationship between the ordinary statistical mechanics[18, 19] and the Conformal Perturbation theory[20] is established. As a byproduct, for an "external" (or, probably more adequately, "foreign") particle with an arbitrary charge put into the log-gas, a relation between its excess chemical potential and the one-point expectations of the exponential boundary field in the sine-Gordon model is derived.

Section 3, devoted to the derivation of full thermodynamics of the 1D TC log-gas, is based on an exact fromula for the above one-point expectations,[21] obtained by using a "reflection" relationship between the 2D boundary Liouville and sinh-Gordon theories.

Section 4 brings a discussion about a possible analytic extension of the acquired results to the collapse region $1 \leqslant \beta \leqslant 2$. In this region, one could attach to the particles a small hard core σ in order to prevent the collapse, then calculate thermodynamic quantities, and at the end take the $\sigma \to 0$ limit. In this limit, while the free energy and the internal energy per particle diverge, the specific heat, truncated correlation functions, etc., are expected to remain finite.[22] An analytic continuation of the formula for the specific heat (at constant volume) predicts an infinite sequence of phase transitions from the conducting plasma phase $\beta \leqslant 1$ to the insulator region $\beta \geqslant 2$. In two dimensions, such a phenomenon was predicted in ref. 23 but later denied in ref. 24.

Whenever possible, the results are checked by a systematic small-β (high-temperature) expansion, using a renormalized Mayer expansion technique developed in refs. 17, 25, and 26 (see Appendix), and close to the collapse $\beta = 1$ point, by applying arguments in the spirit of an independent-pair approximation,[22] which assumes a dominant contribution from almost collapsed positive-negative pairs of charges to the configurational integral.

2. MAPPING ONTO THE BOUNDARY SINE-GORDON THEORY

Let us first examine a general TC plasma of $q = \pm 1$ charges in an infinite 2D space of points $\mathbf{r} = (x, y)$. We will work in the grand canonical ensemble, with position-dependent fugacities $z_+(\mathbf{r})$ and $z_-(\mathbf{r})$ of the positive and negative particles, respectively. In infinite space, $-\Delta/(2\pi)$ is the inverse operator of the Coulomb potential $-\ln(|\mathbf{r}|)$. Using the standard procedure (see, e.g., ref. 27), the grand partition function Ξ of the plasma at inverse temperature β, considered as the functional of particle fugacities, can be turned into

$$\Xi[z_+, z_-] = \frac{\int \mathcal{D}\phi \exp[\int \mathrm{d}^2 r (\frac{1}{4\pi} \phi \, \Delta\phi + z_+(\mathbf{r}) \, \mathrm{e}^{\mathrm{i}\sqrt{\beta}\,\phi} + z_-(\mathbf{r}) \, \mathrm{e}^{-\mathrm{i}\sqrt{\beta}\,\phi})]}{\int \mathcal{D}\phi \exp(\int \mathrm{d}^2 r \, \frac{1}{4\pi} \phi \, \Delta\phi)} \tag{2.1}$$

Here, $\phi(\mathbf{r})$ is a real scalar field, $\int \mathcal{D}\phi$ denotes the functional integration over this field and the fugacities are renormalized by a self-energy term. The consideration of

$$z_q(\mathbf{r}) = z_q(x) \, \delta(y) \tag{2.2}$$

confines the charges to the x-line,

$$\Xi[z_+, z_-] = \frac{\int \mathcal{D}\phi \exp\{\int \mathrm{d}^2 r (\frac{1}{4\pi} \phi \, \Delta\phi) + \int_{-\infty}^{\infty} \mathrm{d}x [z_+(x) \, \mathrm{e}^{\mathrm{i}\sqrt{\beta}\,\phi_B} + z_-(x) \, \mathrm{e}^{-\mathrm{i}\sqrt{\beta}\,\phi_B}]\}}{\int \mathcal{D}\phi \exp(\int \mathrm{d}^2 r \, \frac{1}{4\pi} \phi \, \Delta\phi)} \tag{2.3}$$

where $\phi_B(x) \equiv \phi(x, y = 0)$ is the boundary field.

To reformulate the field theory (2.3) as a boundary problem, in formal analogy with refs. 28 and 29, one introduces two new fields

$$\phi_e(x, y) = \frac{1}{\sqrt{2}} [\phi(x, y) + \phi(x, -y)] \tag{2.4a}$$

$$\phi_o(x, y) = \frac{1}{\sqrt{2}} [\phi(x, y) - \phi(x, -y)] \tag{2.4b}$$

defined only in the upper half-plane $y \geqslant 0$. The even field has a Neumann boundary condition $\partial_y \phi_e|_{y=0} = 0$ and the odd field has a Dirichlet boundary condition $\phi_o|_{y=0} = 0$. It holds

$$\int \mathrm{d}^2 r \, \phi \, \Delta\phi = \int_{y>0} \mathrm{d}^2 r (\phi_e \, \Delta\phi_e + \phi_o \, \Delta\phi_o) \tag{2.5}$$

The odd field, contributing only by its free-field part $\phi_o\, \Delta\phi_o$, disapears from (2.3) by numerator-denominator cancelation. By integration per partes, the term $\phi_e\, \Delta\phi_e$ can be rewritten as $-(\nabla\phi_e)^2$, with a vanishing contribution from the boundary. Considering that $\phi_B(x) = \phi_e(x, y=0)/\sqrt{2}$, and renaming then ϕ_e as ϕ, (2.3) transforms to

$$\Xi[z_+, z_-] = \frac{\int \mathscr{D}\phi \, \exp(-S_{sG}[z_+, z_-])}{\int \mathscr{D}\phi \, \exp(-S_{sG}[0, 0])} \tag{2.6}$$

with the action

$$S_{sG}[z_+, z_-] = \int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}y \, \frac{1}{4\pi} (\nabla\phi)^2 - \int_{-\infty}^{\infty} \mathrm{d}x [z_+(x)\, \mathrm{e}^{\mathrm{i}b\phi_B} + z_-(x)\, \mathrm{e}^{-\mathrm{i}b\phi_B}] \tag{2.7a}$$

$$\beta = 2b^2 \tag{2.7b}$$

where again $\phi_B(x) \equiv \phi(x, y=0)$ and the ϕ-field has the Neumann boundary condition. For uniform and equivalent charge fugacities, $z_+(x) = z_-(x) = z$, one gets the boundary sine-Gordon model with zero bulk mass,

$$S_{sG}(z) = \int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}y \, \frac{1}{4\pi} (\nabla\phi)^2 - 2z \int_{-\infty}^{\infty} \mathrm{d}x \cos(b\phi_B) \tag{2.8}$$

The sine-Gordon representation of the multi-particle densities can be obtained from the functional generator (2.6), (2.7) in a straightforward way. The density of particles of one sign is

$$n_q = z_q \left. \frac{\delta \ln \Xi}{\delta z_q(x)} \right|_{z_q(x)=z} = z_q \langle \mathrm{e}^{\mathrm{i}qb\phi_B} \rangle_{sG} \tag{2.9}$$

where $\langle \cdots \rangle_{sG}$ denotes the averaging over the sine-Gordon action (2.8). Here, although the charge symmetry is considered, i.e., $n_+ = n_- = n/2$ (n is the total number density of particles), we leave the q-subscript in order to make transparent identities like $\langle \mathrm{e}^{\mathrm{i}b\phi_B} \rangle_{sG} = \langle \mathrm{e}^{-\mathrm{i}b\phi_B} \rangle_{sG}$. This identity is a special case of a general symmetry relation

$$\langle \mathrm{e}^{\mathrm{i}a\phi_B} \rangle_{sG} = \langle \mathrm{e}^{-\mathrm{i}a\phi_B} \rangle_{sG}, \qquad a \text{ arbitrary} \tag{2.10}$$

which results from the invariance of the sine-Gordon action (2.8) with respect to the transformation $\phi \to -\phi$. The pair distribution function $g_{q,q'}(x, x')$ is given by

$$g_{q,q'}(|x-x'|) = \left(\frac{z_q}{n_q}\right)\left(\frac{z_{q'}}{n_{q'}}\right) \frac{1}{\Xi} \frac{\delta^2 \Xi}{\delta z_q(x)\, \delta z_{q'}(x')}\bigg|_{z_q(x)=z}$$

$$= \left(\frac{z_q}{n_q}\right)\left(\frac{z_{q'}}{n_{q'}}\right) \langle \mathrm{e}^{\mathrm{i}qb\phi_B(x)} \mathrm{e}^{\mathrm{i}q'b\phi_B(x')} \rangle_{sG} \tag{2.11}$$

and so on.

In statistical mechanics, the short-distance behaviour of the pair distribution function $g_{q,q'}(|x-x'|)$ is dominated by the Boltzmann factor of the pair potential,

$$g_{q,q'}(x, x') \sim C_{q,q'}\, |x-x'|^{\beta q q'} \qquad \text{as} \quad |x-x'| \to 0 \tag{2.12}$$

(provided β is small enough; see below). The prefactors $C_{q,q'}$ are related to a free energy difference.[18, 19] For $q' = -q$, one has

$$C_{q,-q} = \exp[\beta(\mu_+^{\mathrm{ex}} + \mu_-^{\mathrm{ex}})] \tag{2.13}$$

where μ_q^{ex} is the excess chemical potential of species q. Strictly speaking, relations (2.12) and (2.13) are valid only for $\beta < 1$; at $\beta = 1$, due to the collapse of positive-negative pairs of charges, $\mu_\pm^{\mathrm{ex}} \to \infty$. On the other hand, for the truncated two-body densities $n_{q,q'}^{(\mathrm{T})}(x, x') = n_q n_{q'} [g_{q,q'}(x, x') - 1]$, the short-distance asymptotics analogous to that given by Eq. (2.12) with $q' = -q$ takes place also for $\beta > 1$. For $q' = q$, one has

$$C_{q,q} = \exp[\beta(2\mu_q^{\mathrm{ex}} - \mu_{2q}^{\mathrm{ex}})] \tag{2.14}$$

Here, we use an extended definition of the excess chemical potential for "foreign" ions with arbitrary charges, put into the underlying electrolyte: μ_Q^{ex} = reversible work which has to be done in order to bring a foreign particle of charge Q from infinity into the bulk interior of the 1D TC plasma of unit $\pm$ charges (the consequent breaking of the system neutrality has a negligible effect in the thermodynamic limit). This quantity is of evident importance in chemistry. The interaction Boltzmann factor of the Q charge and an opposite unit charge at distance x, $x^{-\beta|Q|}$, is integrable at small x if and only if $|Q|\beta < 1$. The stability region for μ_Q^{ex} therefore is expected to be $\beta < 1/|Q|$ with μ_Q^{ex} diverging just at $\beta = 1/|Q|$. As a consequence, relations (2.12) and (2.14) are valid only for $\beta < 1/2$. For $\beta > 1/2$, the two-point correlator $\propto |x-x'|^{1-\beta}$; the analysis of this change in the short-distance expansion goes beyond the scope of the present work. With regard to the thermodynamic relation

$$\ln\left(\frac{z_q}{n_q}\right) = \beta\mu_q^{\text{ex}} \tag{2.15}$$

we conclude that, for $q = \pm$, it holds

$$\langle e^{iqb\phi_B}\rangle_{sG} = \exp(-\beta\mu_q^{\text{ex}}) \tag{2.16}$$

and

$$\langle e^{iqb\phi_B(x)} e^{-iqb\phi_B(x')}\rangle_{sG} \sim |x-x'|^{-2b^2} \qquad \text{as} \quad |x-x'| \to 0 \tag{2.17a}$$

$$\langle e^{iqb\phi_B(x)} e^{iqb\phi_B(x')}\rangle_{sG} \sim e^{-\beta\mu_{2q}^{\text{ex}}} |x-x'|^{2b^2} \qquad \text{as} \quad |x-x'| \to 0 \tag{2.17b}$$

where we have used the equality $\beta = 2b^2$, see formula (2.7b).

In quantum field theory, the sine-Gordon model (2.8) can be regarded as a conformal field theory perturbed by the boundary cos-field. The short distance expansions for multi-point correlation functions are then obtainable by using the Operator Product expansions (OPE).[30] The vacuum one-point expectations, in our case $\langle \exp(ia\phi_B)\rangle_{sG}$ with an arbitrary value of a, are the basic objects of the OPE scheme which contain the whole nonperturbative information about the system. For the product of two primary fields $e^{ia_1\phi}(x_1)e^{ia_2\phi}(x_2)$, the OPE has the form

$$e^{ia_1\phi_B}(x_1)e^{ia_2\phi_B}(x_2) = \sum_{n=-\infty}^{\infty} [C_{a_1,a_2}^{n,0}(|x_1-x_2|)\, e^{i(a_1+a_2+nb)\,\phi_B}(x_2) + \cdots] \tag{2.18}$$

where the coefficients C are computable within the Conformal Perturbation theory.[20] By successive application of (2.18) the short-distance behaviour of any multi-point correlation function of the exponential field can be reduced to one-point functions. For the present model, the leading short-distance term of the two-point function reads

$$\langle e^{iqb\phi_B(x)} e^{iq'b\phi_B(x')}\rangle_{sG} \sim \langle e^{i(q+q')\,b\phi_B}\rangle_{sG}\, |x-x'|^{2qq'b^2} \qquad \text{as} \quad |x-x'| \to 0 \tag{2.19}$$

For $q' = -q$, formula (2.17a) is reproduced. For $q' = q$, the combination of (2.17b) and (2.19) implies the relationship

$$\exp(-\beta\mu_{2q}^{\text{ex}}) = \langle e^{i2qb\phi_B}\rangle_{sG} \tag{2.20}$$

The above formalism generalizes straightforwardly to Q-particle distribution functions (Q positive integer), with the result

$$\exp(-\beta\mu_Q^{\text{ex}}) = \langle e^{iQb\phi_B}\rangle_{sG} \tag{2.21}$$

Eqs. (2.16) and (2.20) are the lowest $Q=1$ and $Q=2$ cases, respectively, of this formula. We expect the validity of relation (2.21) also for noninteger values of Q. An analogous formula can be derived for the infinite (unconfined) 2D TC plasma.

3. THERMODYNAMICS

The 2D boundary sine-Gordon theory with the action (2.8) has a discrete symmetry $\phi \to \phi + 2\pi n/b$ with any integer n. In the domain $0 < b^2 < 1$ this symmetry is spontaneously broken and the theory has infinitely many ground states $|0_n\rangle$, characterized by the associate expectation values of the field ϕ, $\langle \phi \rangle_n = 2\pi n/b$. One has to choose one of these ground states, say $|0_0\rangle$, and consider $\langle \cdots \rangle_{sG} \equiv \langle \cdots \rangle_0$. The parameter z, which renormalizes multiplicatively, gets a precise meaning when one fixes the normalization of the field $\cos(b\phi_B)$. The conformal normalization,[21] based on the OPE scheme (2.18) and consistent with formula (2.17a) derived via the 1D TC log-gas, corresponds to the short-distance limit of the two-point function

$$\langle \mathrm{e}^{\mathrm{i}a\phi_B(x)} \mathrm{e}^{-\mathrm{i}a\phi_B(x')} \rangle_{sG} \sim |x-x'|^{-2a^2} \qquad \text{as} \quad |x-x'| \to 0 \tag{3.1}$$

Under normalization (3.1), the expectation value of the exponential boundary field was obtained in ref. 21,

$$\langle \mathrm{e}^{\mathrm{i}a\phi_B} \rangle_{sG} = \left[\frac{2^{b^2} \pi z}{\Gamma(b^2)} \right]^{a^2/(1-b^2)} \exp\left\{ \int_0^\infty \frac{\mathrm{d}t}{t} \left[\frac{(\mathrm{e}^t - 1 + \mathrm{e}^{t(1-b^2)} + \mathrm{e}^{-b^2 t}) \sinh^2(abt)}{2\sinh(b^2 t)\sinh(t)\sinh((1-b^2)t)} - a^2 \left(\frac{1}{\sinh((1-b^2)t)} + \mathrm{e}^{-t} \right) \right] \right\} \tag{3.2}$$

where $|\mathrm{Re}\, 2a| < 1/b$. This result was derived under assumption of a "reflection" relationship between the 2D boundary Liouville and sinh-Gordon theories (with zero bulk mass), and the consequent analytic continuation of the latter theory in the b-parameter to the boundary sine-Gordon model. For the case of special interest $a=b$, using the integral representation of the Gamma function

$$\ln \Gamma(x) = \int_0^\infty \frac{\mathrm{d}t}{t} \mathrm{e}^{-t} \left[x - 1 + \frac{\mathrm{e}^{-(x-1)t} - 1}{1 - \mathrm{e}^{-t}} \right], \qquad \mathrm{Re}\, x > 0 \tag{3.3}$$

after some algebra, relation (3.2) takes a simpler form

$$\langle e^{ib\phi_B}\rangle_{sG} = \frac{1}{4\pi^{3/2}(1-b^2)\,z}\,\Gamma\left(\frac{1-2b^2}{2-2b^2}\right)\Gamma\left(\frac{b^2}{2-2b^2}\right)\left[\frac{2\pi z}{\Gamma(b^2)}\right]^{1/(1-b^2)} \tag{3.4}$$

On account of the symmetry relation (2.10), it holds

$$\langle e^{ib\phi_B}\rangle_{sG} = \frac{1}{2}\frac{\partial}{\partial z}\lim_{L\to\infty}\frac{1}{L}\ln \Xi_L \tag{3.5}$$

where L is the length of the system. Since $\Xi_L(z=0)=1$, one has

$$\lim_{L\to\infty}\frac{1}{L}\ln \Xi_L = \frac{1}{2\pi^{3/2}}\,\Gamma\left(\frac{1-2b^2}{2-2b^2}\right)\Gamma\left(\frac{b^2}{2-2b^2}\right)\left[\frac{2\pi z}{\Gamma(b^2)}\right]^{1/(1-b^2)} \tag{3.6}$$

Relating the sine-Gordon parameters with those of the TC log-gas, $\beta = 2b^2$ from (2.7b) and $n/(2z) = \langle \exp(ib\phi_B)\rangle_{sG}$ according to (2.9), formulae in the above paragraph imply the explicit $n-z$ relation for the plasma:

$$\frac{n^{1-\beta/2}}{z} = \frac{2\pi}{\Gamma(\beta/2)}\left[\frac{1}{2\pi^{3/2}(1-\beta/2)}\,\Gamma\left(\frac{1-\beta}{2-\beta}\right)\Gamma\left(\frac{\beta/2}{2-\beta}\right)\right]^{1-\beta/2} \tag{3.7}$$

The small-β expansion of the rhs of (3.7) reads

$$\frac{n^{1-\beta/2}}{z} = 2\beta^{\beta/2}\exp\left\{(C+\ln\pi)\frac{\beta}{2}+\frac{1}{12}\left[\pi^2+7\zeta(3)\right]\left(\frac{\beta}{2}\right)^3 + \frac{1}{12}\left[\pi^2+6\zeta(3)\right]\left(\frac{\beta}{2}\right)^4 + O(\beta^5)\right\} \tag{3.8}$$

where C is the Euler's constant and ζ is the Riemann's zeta function. Formula (3.8) is checked in the Appendix by using a renormalized Mayer expansion in density. For fixed z, relation (3.7) tells us that the particle density n exhibits the expected collapse singularity as $\beta \to 1^-$:

$$n \sim \frac{4z^2}{1-\beta} \tag{3.9}$$

The form of this singularity can be reproduced indirectly by using a perfect-screening sum rule for the one-body densities,

$$n_q = \int dx\left[n^{(T)}_{q,-q}(x) - n^{(T)}_{q,q}(x)\right] \tag{3.10}$$

where the truncated two-body densities $n^{(T)}_{q,q'}(x,x') = n_q n_{q'}[g_{q,q'}(x,x') - 1]$. For $\beta \to 1^-$, the integral in (3.10) is dominated by the short-distance behaviour of $n^{(T)}_{q,-q}(x) \sim z^2 |x|^{-\beta}$ [see relations (2.12), (2.13) and (2.15)]. Then,

$$\frac{n}{2} = \int_{-\lambda}^{\lambda} \mathrm{d}x \, \frac{z^2}{|x|^\beta} = \frac{2z^2}{1-\beta} \lambda^{1-\beta} = \frac{2z^2}{1-\beta} + O(1) \tag{3.11}$$

where λ is a length over which the Coulomb interaction is screened by the system, and the singular behaviour (3.9) is justified.

To get the full thermodynamics of the 1D TC log-gas, we pass from the grandcanonical to the canonical ensemble via the Legendre transformation

$$F_L(\beta, N) = \Omega_L + \mu N \tag{3.12}$$

Here, the total particle number $N = nL$, the chemical potential

$$\mu(\beta, n) = \frac{1}{\beta} \ln z(\beta, n) \tag{3.13}$$

and the grand potential Ω is defined by

$$-\beta \Omega_L = \ln \Xi_L = L\left(1 - \frac{\beta}{2}\right) n \tag{3.14}$$

where we have combined Eqs. (3.4) and (3.6). Note that the thermodynamic relation $\ln \Xi_L = \beta P L$ (P is the pressure) implies the well-known equation of state

$$\beta P = \left(1 - \frac{\beta}{2}\right) n \tag{3.15}$$

The dimensionless specific free energy f, $f = \beta F_L / N$, is then written as

$$\begin{aligned} f(\beta, n) = {} & \left(1 - \frac{\beta}{2}\right)\left[-1 + \ln n + \ln\left(1 - \frac{\beta}{2}\right)\right] - \frac{\beta}{2} \ln 2 + \frac{1}{2}\left(1 - \frac{3\beta}{2}\right) \ln \pi \\ & + \ln \Gamma\left(\frac{\beta}{2}\right) - \left(1 - \frac{\beta}{2}\right)\left[\ln \Gamma\left(\frac{1-\beta}{2-\beta}\right) + \ln \Gamma\left(\frac{\beta/2}{2-\beta}\right)\right] \end{aligned} \tag{3.16}$$

According to the elementary thermodynamics, the (excess) internal energy per particle, $u^{\mathrm{ex}} = \langle E \rangle$, and the (excess) specific heat at constant volume per particle, $c_V^{\mathrm{ex}} = C_V^{\mathrm{ex}} / N$, are given by

$$u^{\mathrm{ex}} = \frac{\partial}{\partial \beta} f(\beta, n) \tag{3.17a}$$

$$\frac{c_V^{\mathrm{ex}}}{k_B} = -\beta^2 \frac{\partial^2}{\partial \beta^2} f(\beta, n) \tag{3.17b}$$

For c_V^{ex}, one gets explicitly

$$\frac{c_V^{\mathrm{ex}}}{k_B} = \frac{\beta^2}{4(2-\beta)^3} \left\{ 2\psi'\left(\frac{1-\beta}{2-\beta}\right) + (\beta-2)^3 \psi'\left(\frac{\beta}{2}\right) - 2(\beta-2)^2 + 2\psi'\left(\frac{\beta/2}{2-\beta}\right) \right\} \tag{3.18}$$

where $\psi(x) = \mathrm{d}[\ln \Gamma(x)]/\mathrm{d}x$ is the psi function. The series representation of its first derivative reads

$$\psi'(x) = \sum_{j=0}^{\infty} \frac{1}{(x+j)^2} \tag{3.19}$$

The expansion of c_V^{ex}/k_B around the collapse point $\beta \to 1^-$ results in the Laurent series

$$\frac{c_V^{\mathrm{ex}}}{k_B} = \frac{1}{2(1-\beta)^2} - \frac{3}{2(1-\beta)} + \left(\frac{3}{2} + \frac{5\pi^2}{24}\right) + O(1-\beta) \tag{3.20}$$

The leading term of this series can be reproduced by regarding that, under assumption (3.11),

$$f(\beta, n) \sim \tfrac{1}{2} \ln(1-\beta) - \tfrac{1}{2}(1-\beta) \ln \lambda \qquad \text{as} \quad \beta \to 1^- \tag{3.21}$$

and then applying (3.17b).

As concerns the excess chemical potential of a foreign particle of charge Q in the plasma, setting $a = bQ$ and $b^2 = \beta/2$ in (3.2), relation (2.21) gives

$$-\beta\mu_Q^{\mathrm{ex}} = \frac{\beta Q^2}{2-\beta} \ln\left[\frac{2^{\beta/2}\pi z}{\Gamma(\beta/2)}\right] + \int_0^\infty \frac{\mathrm{d}t}{t} \left[\frac{(\mathrm{e}^t - 1 + \mathrm{e}^{t(1-\beta/2)} + \mathrm{e}^{-\beta t/2}) \sinh^2(Q\beta t/2)}{2 \sinh(\beta t/2) \sinh(t) \sinh((1-\beta/2)t)} - \frac{\beta Q^2}{2} \left(\frac{1}{\sinh((1-\beta/2)\, t)} + \mathrm{e}^{-t} \right) \right] \tag{3.22}$$

In the limit of large t, the first integrand behaves like $(1/t) \exp[(\beta|Q|-1)t]$, and therefore the integral is finite if $\beta < 1/|Q|$. As was argued in the previous section, this is the true stability region for μ_Q^{ex}. For $|Q|<1$, μ_Q^{ex} remains finite also in a subspace of the collapse region, namely in the interval $1<\beta<1/|Q|$. Everything what was said above holds only when $\beta<2$. Approaching the conductor-insulator phase transition at point $\beta=2$, the first term on the rhs of (3.22) diverges for an arbitrary Q.

4. DISCUSSION

All results in this paper were derived under the assumption that the system is stable against collapse, i.e., in the range of inverse temperatures $\beta<1$. The only exception is represented by the excess chemical potential μ_Q^{ex} of a "foreign" Q-charged particle embedded in the plasma: if $|Q|<1$, relation (3.22) applies also to the collapse region, up to $\beta=1/|Q|$. A natural question arises whether there is a possibility to make an analytic continuation of the explicit result for some bulk quantity into the collapse region $1<\beta<2$. The best candidate for such a continuation is the specific heat, formula (3.18), which might be in the sense explained in the Introduction a well defined finite quantity also in the collapse region.

Before going further, let us recall the work of Gallavotti and Nicoló[23] concerning the infinite 2D TC Coulomb gas of unit $\pm$ charges. In two dimensions, the particle collapse occurs at $\beta=2$ and the Kosterlitz–Thouless phase transition (for a small, but nonzero, hard core of particles) takes place around $\beta=4$. By studying the Mayer series of the specific grand potential in *fugacity* it was proven in ref. 23 that each term of the series converges in the insulator region $\beta>4$. For $\beta\leqslant 4$, the existence of infinitely many thresholds

$$2D: \quad \beta_N = 4\left(1-\frac{1}{2N}\right), \qquad N=1,2,\ldots \tag{4.1}$$

was observed: only the Mayer series' coefficients up to order $2N$ are finite if $\beta>\beta_N$. Points $\{\beta_N\}$ were conjectured to correspond to a sequence of transitions from the pure multipole insulating phase ($\beta>4$) to the conducting plasma phase ($\beta<2$) via an infinite number of intermediate phases. Although the mathematics used in ref. 23 was quite complicated, there exists a simple explanation of the above findings.[32] For a neutral choice of N positive and N negative charges in a disk of radius R, the most relevant contribution to the coefficient of the z^{2N} term comes from the complete-star integral, after approximating the Mayer function by $\exp(-\beta v)$,

$$\frac{1}{\pi R^2}\int_\sigma^R \prod_{i=1}^{N} \mathrm{d}^2 r_i\, \mathrm{d}^2 r_i' \, \frac{\prod_{i<j} |\mathbf{r}_i-\mathbf{r}_j|^\beta \, |\mathbf{r}_i'-\mathbf{r}_j'|^\beta}{\prod_{i,j} |\mathbf{r}_i-\mathbf{r}_j'|^\beta} \tag{4.2}$$

where σ is a small hard-core diameter and $\mathbf{r}$ ($\mathbf{r}'$) denote spatial coordinates of positive (negative) charges. Rescaling $\mathbf{r}_i \to R s_i$ and $\mathbf{r}'_i \to R s'_i$, one gets (4.2) $= R^{N(4-\beta)-2} \times f(\sigma/R)$ which diverges as $R \to \infty$ just when $\beta < \beta_N$. In two dimensions, the formula for c_V^{ex}/k_B in the region $\beta < 2$ is presented in ref. 17, relation (56). The only source of the singularities of c_V^{ex}/k_B, extended into the region $2 < \beta < 4$, is the term $\propto \sin^{-2}(\pi\beta/(4-\beta))$, which gives a double pole at

$$\bar{\beta}_N = 4\left(1 - \frac{1}{N+1}\right) \qquad N = 1, 2, \ldots \tag{4.3}$$

For $N = 1, 3, 5, \ldots$, these singular points coincide with the ones in (4.1). However, there are additional divergencies of c_V^{ex}/k_B for $N = 2, 4, 6, \ldots$ when $c_V^{\text{ex}}/k_B \to -\infty$, which is an unacceptable thermodynamic behaviour. Therefore, in 2D, the analytic extension of the formula for c_V^{ex}/k_B is meaningless, what supports the arguments of Fisher *et al.*[24] indicating the total absence of any intermediate phase at nonzero particle density.

The situation is different in 1D. The dominant configuration integral of the z-series at the $2N$th order

$$\frac{1}{L} \int_\sigma^L \prod_{i=1}^{N} \mathrm{d}x_i \, \mathrm{d}x'_i \, \frac{\prod_{i<j} |x_i - x_j|^\beta \, |x'_i - x'_j|^\beta}{\prod_{i,j} |x_i - x'_j|^\beta} \tag{4.4}$$

diverges for the line length $L \to \infty$ when $\beta < \beta_N$ with

$$1D\colon \quad \beta_N = 2\left(1 - \frac{1}{2N}\right), \qquad N = 1, 2, \ldots \tag{4.5}$$

The singularities of c_V^{ex}/k_B, given by formula (3.18), originate in the region $1 < \beta < 2$ exclusively from the term $\psi'((1-\beta)/(2-\beta))$. According to (3.19), $\psi'(x)$ has second-order poles at $x = 1 - N$ ($N = 1, 2, \ldots$). The corresponding $\bar{\beta}_N$ coincide just with β_N, so the analytic continuation of (3.18) into the collapse region $1 < \beta < 2$ reproduces exactly the singularities suggested by Gallavotti and Nicoló. Moreover, c_V^{ex}/k_B is always *positive* in the underlying region $1 < \beta < 2$. It is therefore tempting to conjecture that the multipole intermediate phases might exist in 1D.

The last remark concerns the generalization of the relation (3.11) to the case of a small hard core σ:

$$\frac{n}{2} = 2 \int_\sigma^\lambda \mathrm{d}x \, \frac{z^2}{|x|^\beta} = \frac{2z^2}{1-\beta} \left(\lambda^{1-\beta} - \sigma^{1-\beta}\right) \tag{4.6}$$

with $\lambda > \sigma$. Then, around $\beta = 1$, the free energy

$$f \sim \frac{1}{2} \ln \left(\frac{1-\beta}{\lambda^{1-\beta} - \sigma^{1-\beta}} \right) \tag{4.7}$$

is well defined for $\beta \to 1^-$ as well as $\beta \to 1^+$. Applying (3.17b) and taking the $\sigma \to 0$ limit, one finds

$$\frac{c_V^{\mathrm{ex}}}{k_B} \sim \frac{1}{2(1-\beta)^2} \qquad \text{for both } \beta \to 1^- \text{ and } \beta \to 1^+ \tag{4.8}$$

i.e., the leading singular term of the expansion (3.20) admits an analytic continuation from $\beta < 1$ to the $\beta > 1$ region. This fact supports the above conjecture.

We hope to motivate numerical simulations of the model under consideration.

Finding an integrable TC Coulomb gas with some short-range regularization of the interaction Coulomb potential might be a realistic task in one dimension.

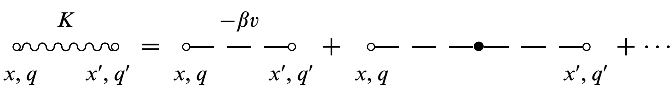
APPENDIX

We consider the 1D TC log-gas confined to a straight line of size $L \to \infty$. A pair of $(\pm)$-charged particles i, j interacts via

$$v(x_i, q_i \mid x_j, q_j) = q_i q_j v(x_i, x_j) \tag{A.1}$$

where the Coulomb potential $v(x, x')$ is defined by (1.3). The supposed equality of the species fugacities $z_+ = z_- = z$ corresponds to a neutral system with homogeneous particle densities $n_+ = n_- = n/2$.

The renormalized Mayer expansion in density (for details, see refs. 17, 25, and 26) is based on the expansion of each Mayer function in the inverse temperature β, and on the consequent series elimination of two-coordinated field circles between every couple of three- or more-coordinated field circles; by coordination of a circle we mean its bond coordination. The renormalized bonds are given by



(A.2)

where, besides the integration over spatial coordinate x of a field (black) $n_q(x)$-circle, the summation over charge q-states at this vertex is assumed as well. For the interaction under consideration (A.1), the renormalized bonds exhibit the same charge-dependence,

$$K(x, q \mid x', q') = qq' K(x, x') \tag{A.3a}$$

where $K(x, x') = K(|x - x'|)$ is given by

$$K(x) = -\frac{\beta}{2} \int_{-\infty}^{\infty} \mathrm{d}k \, \frac{\exp(ikx)}{|k| + \pi n \beta} \tag{A.3b}$$

By a simple rescaling of integration variable k, K is shown to exhibit a special scaling form

$$K(x) = -\beta \bar{K}(\pi n \beta \, |x|) \tag{A.4a}$$

$$\bar{K}(x) = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}k \, \frac{\exp(ikx)}{|k| + 1} \tag{A.4b}$$

The small-x expansion of K reads[31]

$$K(x) \sim \beta[\ln(\pi n \beta \, |x|) + C - \tfrac{1}{2} \pi^2 n \beta \, |x| + \cdots] \tag{A.5}$$

where C is the Euler's constant. At asymptotically large-x distances, by using twice integration per partes it can be shown that K decays algebraically as follows

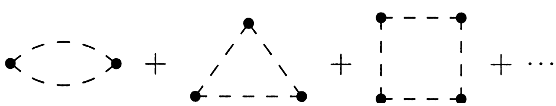
$$K(x) \sim -\frac{\beta}{(\pi n \beta x)^2} \tag{A.6}$$

which exhibits the poor screening properties of log-gases in 1D.[1,2]

The procedure of bond-renormalization transforms the ordinary Mayer representation of the dimensionless (minus) excess Helmholtz free energy, denoted as $\Delta[n]$, into

$$\Delta[n] = \bullet\!-\;-\;-\!\bullet + D^{(0)}[n] + \sum_{s=1}^{\infty} D^{(s)}[n], \tag{A.7a}$$

where

$D^{(0)} =$  (A.7b) is the sum of all unrenormalized ring diagrams and $\{D^{(s)}\}$ represents the set of all remaining completely renormalized graphs; the first few $D^{(s)}$-graphs are drawn in the sketch (11) of ref. 17. The first term on the rhs of (A.7a) is fixed to zero by charge neutrality. The second term $D^{(0)}$, Eq. (A.7b), is expressible as follows[17]

$$D^{(0)} = \frac{L}{2} \int_0^n \mathrm{d}n' \lim_{x \to 0} \left[K(x, n') + \beta v(x) \right] \tag{A.8}$$

With respect to (A.5), one gets

$$\frac{D^{(0)}}{L} = \frac{\beta}{2} (n \ln n - n) + \frac{\beta n}{2} \left[C + \ln(\pi\beta) \right] \tag{A.9}$$

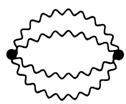
As concerns the completely renormalized graphs, due to the $\pm$ charge symmetry, only those $D^{(s)}$ are nonzero whose all vertices have an even bond coordination. The scaling form of the dressed bond K, formula (A.4), permits us to perform the n- and β-classification of a nonzero diagram $D^{(s)}$, composed of N_s vertices (each bringing the factor n) and L_s bonds. Each dressed bond contributes by the factor $-\beta$ and enforces the substitution $x' = x\pi n\beta$ which manifests itself as the factor $1/(\pi n \beta)$ for each field-circle integration $\int \mathrm{d}x$. Since there are $(N_s - 1)$ independent field-circle integrations in $D^{(s)}$, one concludes that

$$\frac{D^{(s)}}{L} = n\beta^{L_s - N_s + 1} d_s \tag{A.10a}$$

where d_s is the number

$$d_s = \frac{D^{(s)}(n = 1, \beta = 1)}{L} \tag{A.10b}$$

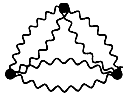
The first nonzero diagram from the sketch (11) of ref. 17 is

$D^{(2)} =$  (A.11a)

It contributes to the β^3 order, with

$$d_2 = \frac{1}{2!\,4!} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\pi} \bar{K}^4(x) \tag{A.11b}$$

In the next β^4 order, only diagram

$$D^{(6)} =$$  (A.12a)

survives, and

$$d_6 = \frac{1}{3!\,(2!)^3} \int_{-\infty}^{\infty} \frac{\mathrm{d}x_1}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}x_2}{\pi}\, \bar{K}^2(x_1)\, \bar{K}^2(x_2)\, \bar{K}^2(|x_1 - x_2|)) \qquad \text{(A.12b)}$$

etc. To evaluate d_2 and d_6, we Fourier-transform $\bar{K}^2(x)$:

$$\hat{G}(k) = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}}\, \mathrm{e}^{-\mathrm{i}kx}\, \bar{K}^2(x)$$
$$= \sqrt{2\pi}\, \frac{(1+|k|)}{|k|\,(2+|k|)} \ln(1+|k|) \qquad \text{(A.13)}$$

In terms of $\hat{G}(k)$,

$$d_2 = \frac{1}{2!\,4!\,\pi} \int_{-\infty}^{\infty} \mathrm{d}k\, \hat{G}^2(k) \qquad \text{(A.14a)}$$

$$d_6 = \frac{1}{3!\,(2!)^3\,\pi^2} \sqrt{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k\, \hat{G}^3(k) \qquad \text{(A.14b)}$$

With the aid of the substitution $1+k = \exp(t)$ (k and t positive),

$$d_2 = \frac{1}{12}\frac{1}{2^3}[\pi^2 + 7\zeta(3)] \qquad \text{(A.15a)}$$

$$d_6 = \frac{1}{12}\frac{1}{2^4}[\pi^2 + 6\zeta(3)] \qquad \text{(A.15b)}$$

where ζ is the Riemann's zeta function.

Finally, for $q = \pm$,

$$\ln\left(\frac{n_q}{z}\right) = \ln\left(\frac{n}{2z}\right) = \frac{\partial}{\partial n}\left[\frac{\Delta(n)}{L}\right] \qquad \text{(A.16)}$$

Using results of the previous paragraph, one ends up with

$$\frac{n^{1-\beta/2}}{z} = 2\beta^{\beta/2} \exp\left\{ (C+\ln \pi)\frac{\beta}{2} + \sum_{s=1}^{\infty} d_s \beta^{L_s - N_s + 1} \right\}$$

$$= 2\beta^{\beta/2} \exp\left\{ (C+\ln \pi)\frac{\beta}{2} + \frac{1}{12}[\pi^2 + 7\zeta(3)]\left(\frac{\beta}{2}\right)^3 \right.$$

$$\left. + \frac{1}{12}[\pi^2 + 6\zeta(3)]\left(\frac{\beta}{2}\right)^4 + O(\beta^5) \right\} \qquad \text{(A.17)}$$

in agreement with (3.8).

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